MA3233 Combinatorics & Graphs II

$\underline{\text{order}} := |V| =: n$ } G is (n,m)-graph
$\underline{\text{size}} := |E| =: m$

e is $\underline{\text{incident}}$ to v := $\overset{v}{\bullet}\underline{e}$

$\deg(v)$:= # of edges incident to v.

$\deg_G(v) = |N(v)|$

$N(v) := \{v \in V(G) \mid uv \in E(G)\}$

$N_G(v)$

v is $\underline{\text{isolated}} := \deg(v)=0$

v is $\underline{\text{a leaf}} := \deg(v)=1$

v is $\underline{\text{even}} := \deg(v)$ is even

v is $\underline{\text{odd}} := \deg(v)$ is odd

$\underline{\text{complete}}$ (K_n)

$\underline{\text{empty}}$ $(O_n) := \langle V, \{\}\rangle = G$.

$\underline{\text{Handshaking Lemma}} := \sum_{i=1}^{n} \deg(v_i) = 2m$

↳Cor: # odd vertices is even.

$\underline{e(v)} := \max_{u \in V(G)} d(v,u)$

$\underline{\text{rad}(G)} := \min_{v \in V(G)} \max_{u \in V(G)} d(v,u)$

$\underline{\text{diam}(G)} := \max_{v \in V(G)} \max_{u \in V(G)} d(v,u)$

$\underline{\Delta(G)} := \max_{v \in V} \deg(v)$

$\underline{\delta(G)} := \min_{v \in V} \deg(v)$

G is $\underline{r\text{-regular}} := \Delta(G) = r = \delta(G)$

$\underline{G_1 \text{ is isomorphic to } G_2}$ $(G_1 \cong G_2)$
$:= \exists\, \phi : V(G_1) \to V(G_2)$
s.t. ϕ is bijection and
$uv \in E(G_1) \iff \phi(u)\phi(v) \in E(G_2)$

$G_1 \cong G_2 \iff \overline{G_1} \cong \overline{G_2}$

H is a $\underline{\text{subgraph}}$ of G := $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
↳ If $V(H)=V(G)$ then H is a spanning subgraph of G.

walk := list of adjacent vertices

trail := no edge repeated

path := no vertex repeated.

walk ⊃ trail ⊃ path.

$\underline{u,v \text{ is connected}}$:= there is a u-v walk (it is an equiv. relation)

$\underline{G \text{ is connected}}$:= any $u,v \in V(G)$ is connected.

$\underline{d(u,v)}$:= length of shortest path between u and v.
↳d is a metric.

v is a $\underline{\text{cut vertex}} := c(G) < c(G-v)$

e is a $\underline{\text{bridge}} := c(G) < c(G-e)$

$\underline{c(G)}$:= number of components of G.

$\underline{\text{Thm}}$: Let v be incident to a bridge, then:
v is a cut-vertex $\iff \deg(v) \geq 2$.

$\underline{\text{Cycles}}$ & $\underline{\text{bridges}}$: e is a bridge $\iff$ e is not in any cycle

$\underline{\text{Thm of forests}}$: G is a forest $\iff$ G has no cycle $\iff$ all edges of G are not part of some cycle
$\iff$ all edges of G are bridges.

$\underline{\text{Types of vertices in a forest}}$: { isolated pt
leaf
cut-vertex

$\underline{\text{Thm of trees}}$: G is a tree $\iff \forall u,v \in V(G)$ there is exactly one u-v path in G.

G is a tree $\iff \begin{cases} n = m+1 \\ G \text{ is connected} \end{cases}$
$\iff$ G is connected & no cycle
$\iff$ G is connected & all edges are bridges

$\underline{G \text{ connected} \Rightarrow m \geq n+1}$: PF: If not a tree, keep removing edges until all bridges

Algorithm to find any spanning tree: Do $\underset{\substack{\downarrow \\ \text{can detect} \\ \text{cut vertex}}}{\underline{\text{DFS}}}/\underset{\substack{\downarrow \\ \text{can measure} \\ \text{distance}}}{\underline{\text{BFS}}}$, add edge if it (and recurse) does not form a cycle.

Prims/Kruskals/~~Re~~Reverseremove.

subgraph induced by $S \subseteq V(G)$:= $V(H) = S$ and $E(H) = \{uv \in E(G), u \in S, v \in S\}$

subgraph induced by $X \subseteq E(G)$:= $E(H) = X$ and $V(H) = \{v \in V(G) \mid \exists xy \in X \text{ s.t. } v = x \text{ or } v = y\}$

$G - X := \langle V(H), E(H)\rangle$ where $V(H) = V(G)$ and $E(H) = E(G) \setminus X$

$G - S := \langle V(H), E(H)\rangle$ where $V(H) = V(G) \setminus S$ and $E(H) = \ldots$
(subgraph induced by S)

$$\left(\text{Note: } \overset{\text{set}}{A} \setminus \overset{\text{set}}{B} \qquad \underset{\text{graph}}{G} - \underset{\text{edge set}}{X} \qquad G - S^{\leftarrow \text{vertex set}}\right)$$

$G_1 \cup G_2 := \langle V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)\rangle$

$\underline{\text{Degree sequence}}$: $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ ← typically sorted in descending order.

If $(d_1, \ldots, d_n)$ is graphic then:
- $0 \leq d_i \leq n-1$
- $\sum_i d_i$ is even.

- Can use degree sequences to help find all non-isomorphic graphs of fixed n and m.

Assume $d_1 \geq \cdots \geq d_n$. $(d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphic $\iff$ $(d_1, \ldots, d_n)$ is graphic.
"$\Rightarrow$" is trivial.
"$\Leftarrow$" If $N(v_1) = \{v_2, \ldots, v_{d_1+1}\}$ then we are done. Otherwise, can consider a swapping argument to convert $G - \{v_1\}$ to a graph with desired degrees.

Bipartite Graphs

Graph G s.t. $\exists X \sqcup Y = V(G)$
with no edges within X or Y.

Bipartite $\Leftrightarrow$ no odd cycle.

Multigraph: can have multiple edges with same endpoints,
but still no self-loops.

parallel edges := edges that join the same pair of distinct vertices

Adjacency matrix := a_{ij} = # edges joining vertices i and j, $A(G)$:= adj. mat of G. $A(G)$ is symmetric. Hence all eigenvalues are real, and $A(G)$ is diagonalisable.

i.e. $\exists P : P^{-1}AP \in$ diagonal

$\rightarrow \det(A) = \lambda_1 \cdots \lambda_n$

$\rightarrow tr(A) = \lambda_1 + \cdots + \lambda_n = 0$

$\downarrow$

since $tr(AB) = tr(BA)$

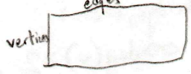
$\rightarrow \sum \lambda_i^2 = tr(A^2) = \sum deg(v_i)$

$= 2m$

$\sum \lambda_i^3 = tr(A^2) = 6 \times$ (# triangles in graph)

Thm. permutation: $G_1 \cong G_2 \Leftrightarrow \exists P$: permutation matrix
s.t. $P^T A(G_2) P = A(G_1)$

Incidence matrix := $b_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is incident with edge } j \\ 0 & \text{otherwise} \end{cases}$, $B(G)$:= incidence mat. of G.



Thm. permutation: $G_1 \cong G_2 \Leftrightarrow \exists P, Q$: permutation matrix
s.t. $P^T B(G_2) Q = B(G_1)$

Thm rank: $rank(B(G)) = n-1$ if G is bipartite; $rank(B(G)) = n$ otherwise

Directed Multigraphs

digraph := directed multigraph

arc := directed edge

v is adjacent from u

u is adjacent to v

e is incident from u

e is incident to v

in-degree  $deg_G^+(v)$

out-degree  $deg_G^-(v)$

Handshaking Lemma: $\sum_{v \in V(G)} deg_G^+(v) = e(G) = \sum_{v \in V(G)} deg_G^-(v)$

Incidence matrix := $b_{ij} \begin{cases} -1 & \text{if } e_j \text{ is incident from } v_i \\ 1 & \text{if } e_j \text{ is incident to } v_i \\ 0 & \text{otherwise.} \end{cases}$, $B(G)$:= incidence mat. of G

walk: $v_0, e_1, v_1, e_2, v_2, \cdots, v_{n-1}, e_n, v_n$

need to specify the edge

Thm rank: $rank(B(G)) = n-1$ for any connected digraph.

if you remove all the directions, it will be connected.

Automorphism: Isomorphism from G to itself

Eulerean Graph (multigraph): has circuit that contains all edges.

Semi-Eulerean Graph (multigraph): has open trail that contains all edges.

Thm: Eulerean $\Rightarrow \forall v \in V(G)$, $deg(v)$ is even.

Thm: Directed Eulerean $\Rightarrow \forall v \in V(G)$, $deg^+(v) = deg^-(v)$.

Thm: $\forall v \in V(G)$, $deg(v)$ is even $\Rightarrow E(G)$ decomposable into cycles

Eulerean graph equivalence:

If G is connected then TFAE:

G has a Eulerean circuit

$\Leftrightarrow$ $\deg(v)$ is even $\forall v \in V(G)$

$\Leftrightarrow$ $E(G)$ decomposable into cycles

$\Leftrightarrow$ each edge of G is contained in an odd number of cycles. $\rightarrow$ Pf:

"$\Leftarrow$" Suppose $\alpha(e)$ are all odd.

$\alpha(v) :=$ # cycles containing v

$\alpha(e) :=$ # cycles containing e

$$\alpha(u) = \frac{1}{2}\sum_{\substack{e\in N(u)\\ \uparrow}} \alpha(e) . \quad \therefore \underset{\underset{\deg(u)}{\shortparallel}}{|N(u)|} \text{ is even.}$$

edges incident to u

"$\Rightarrow$" Can show that in a BFS to find $u \rightarrow v$ trails excluding $u \overset{e}{-} v$ will have an odd number of trails at every BFS steps. Then show that the circuits that are not cycles can be paired off.

Given any Eulerean graph G, removing any edge from G makes a semi-Eulerean graph.

Given any semi-Eulerean graph G, we can add an edge to G to make a Eulerean graph.

Thm: Semi-Eulerean $\Rightarrow$ exactly 2 vertices odd.

Thm: If G has exactly 2 vertices with odd degree, then $E(G)$ decomposable in to exactly one path and any number of cycles.

Semi-Eulerean graph equivalence:

If G is connected then TFAE:

G has a Eulerean open trail

$\Leftrightarrow$ $\deg(v)$ is odd for exactly two $v \in V(G)$

$\Leftrightarrow$ $E(G)$ decomposable into cycles + one path

Algorithms to find Eulerean circuit:

① Repeatedly merge cycles until only one circuit left.

② Start from any vertex, building a trail; but no bridges to be picked unless no other choice.
↖ updated dynamically after picking each edge.

Note: For Semi-Eulerean graphs, just add the missing edge, run the Eulerean graph algorithm, and remove the edge from the Eulerean cycle.

Chinese Postman Problem:

Given a weighted multigraph, find a closed walk with minimum total weight that uses each edge at least once. (Eulerean walk)
↑ positive weights
(or equivalently, duplicate some edges to form a Eulerean graph of minimum total weight)

Algorithm

- arbitrarily pair up odd vertices, and create any path between each pair of vertices
- remove pairs of parallel edges in the paths created (i.e. re-pair those vertices)
- for every cycle with created edges: if $w(\text{duplicated edges in cycle}) > \frac{1}{2} w(\text{cycle})$ then invert duplicated edges in cycle.

} Euleredness of graph preserved, but weight decreases

Furthermore we can prove that any two sets E_1, E_2 of duplicate edges satisfing above algorithm will have $w(E_1) = w(E_2)$.

Algorithm (2):

- Let V be the set of odd vertices, forming a complete graph K, where edge $u-v$ has weight $d_G(u,v)$
- Then run a min weight perfect matching algorithm $O(n^3)$ on it (variant of Edmond blossom)
- Reconstruct the original edges that the chosen matching of K represents.

↑ distance in the original graph.

Hamiltonian Graph: Graph containing a spanning cycle

Semi-Hamiltonian Graph: Graph containing a spanning path
↖ no repeat vertices

If $\deg(v) < 2$ then $\nexists$ spanning cycle

If $\deg(v) = 2$ then both edges must be in spanning cycle.

Necessary conditions for Hamiltonian graph

- $\forall S \subsetneq V(G)$ with $S \neq \phi$, $c(G-S) \leq |S|$
- $\forall S \subsetneq V(G)$ with $S \neq \phi$, $c(G-S) \leq \underset{\substack{\uparrow \\ \text{spanning cycle}}}{c(C-S)} \leq |S|$

Ore's Thm: Given G with order $n \geq 3$:
$\forall$ non-adjacent u, v, $\deg(u) + \deg(v) \geq n$ $\Rightarrow$ G is Hamiltonian → Pf: ① $\forall u, v$: $d(u,v) \leq 2$
② Take any longest path P : then P is a component ↑ can create a cycle from same vertices
③ G must be connected, so $V(P) = V(G)$. (so the cycle is spanning)

Cor: Given G with order $n \geq 3$:
$\delta(G) \geq \frac{n}{2} \Rightarrow G$ is Hamiltonian

Thm: Given G with any order
$\delta(G) \geq \frac{n-1}{2} \Rightarrow G$ is Semi-Hamiltonian → Pf: Add a new vertex that is adjacent to everything else, then use earlier thm, and remove it to get spanning path.

Thm: G Eulerean $\Rightarrow \underset{\substack{\uparrow \\ \text{line graph}}}{L(G)}$ Hamiltonian

Thm: $m \geq \binom{n-1}{2} + 2 \Rightarrow G$ Hamiltonian. → Pf: show that $\forall u, v$, $d(u,v) \leq 2$.

Thm: Given G with order $n \geq 3$:
For any non-adjacent u, v with $\deg(u) + \deg(v) \geq n$: $G + \{uv\}$ is Hamiltonian $\Leftrightarrow$ G is Hamiltonian
Pf ($\Rightarrow$): G has u-v Hamiltonian path. Then use same pf as Ore's thm ②

Closure of G := graph obtained by adding edge $\{uv\}$ whenever $\{uv\} \notin G$ and $\deg(u) + \deg(v) \geq n$. ($Cl(G)$)

Thm: G Hamiltonian $\Leftrightarrow$ $Cl(G)$ Hamiltonian

Thm: Closure is well-defined.

Thm: If G has a degree sequence $(d_1, d_2, \ldots, d_n)$ where $d_1 \leq d_2 \leq \ldots \leq d_n$:
$\left(\forall i < \frac{n}{2}, \ d_i \leq i \Rightarrow d_{n-i} \geq n-i \right) \Rightarrow Cl(G) = K_n \Rightarrow G$ is Hamiltonian → Pf:

Bipartite sufficient condition: If G is bipartite with parts V_1, V_2 : $|V_1| = |V_2| = p$ then : $\delta(G) > \frac{p}{2} \Rightarrow G$ is Hamiltonian.

Travelling Salesman Problem: (on a weighted complete graph)

Approximation algorithms:

① Nearest neighbour: greedy algorithm that chooses the closest unvisited vertex to current vertex each time → no bound on optimality

② Min edge: greedy algorithm that picks shortest edge each time, and adds it to the set if the set remains valid (i.e. a subset of some cycle)
↳ doesn't form a cycle with length $< n$
↳ doesn't cause degree of any vertex to exceed 2.

③ At each iteration find the shortest edge $\{u_i, v_i\}$ where u_i in the selected cycle and v_i not in the selected cycle.
Then extend cycle to include v_i before or after u_i. If it is metric, then it is 2-optimal. (Proof using Prim's algorithm).
(assuming metric property is satisfied)
Cycle length ≤ 2 (length of MST) ≤ 2 (min spanning cycle)

④ Christofides' algorithm: Construct an MST, and run Edmond matching on odd vertizes to get Chinese postman solution, and get a spanning cycle from there.
$\frac{3}{2}$-opt.

Connectivity

v is a cut-vertex := $c(G-v) > c(G)$
e is a bridge := $c(G-e) > c(G)$
$S \subsetneq V(G)$ is a cut of G := $G-S$ is disconnected.
vertex-connectivity of G := $K(G)$:= $\min_S |S|$ where S is a cut of G (for complete graph, $K(K_n) := n-1$)

Cor:
- G disconnected $\Leftrightarrow$ ϕ is a cut of G $\Leftrightarrow$ $K(G) = 0$
- $K(C_n) = 2$
- $K(K_{p,q}) = \min\{p, q\}$ → Pf: since every complete bipartite graph with $p, q \geq 1$ is connected.
- $K(G) = 1 \Leftrightarrow G$ has a cut-vertex

$F \subsetneqq E(G)$ is an edge-cut of G := $G-F$ is disconnected.

edge-connectivity of G := $\lambda(G) := \min_F |F|$ where F is an edge-cut of G (for K_1, $\lambda(K_1) := 0$)

Cor:
- $\lambda(G)=0 \iff G$ is disconnected or $G=K_1$
- $\lambda(G)=1 \iff G$ has a bridge
- $\lambda(C_n)=2$
- $\lambda(P_n)=1$
  ↖ path of length $n\geq 2$.
- $\lambda(T)=1$
  ↖ tree of order $n\geq 2$.

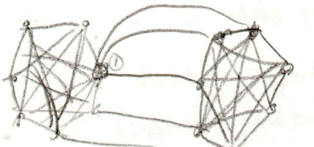
- If F is a min edge cut of G, then $c(G-F)=2$

Thms:
- G Eulerian $\Rightarrow \lambda(G)\geq 2$
- G Hamiltonian $\Rightarrow K(G)\geq 2$
- G connected with $n\geq 2$ $\Rightarrow K(G)-1 \leq K(G-v)$ for any $v\in V(G)$
  $\Rightarrow \lambda(G)-1 \leq \lambda(G-e)$ for any $e\in E(G)$
  $\Rightarrow \lambda(G-e) \leq \lambda(G)$ for any $e\in E(G)$
- $K(G) \leq \lambda(G) \leq \delta(G)$
  ↖ because we can just remove all edges incident to a particular vertex.
  ① show that the edge set F that leads to $\lambda(G)$ will disconnect the graph into two parts G_1 and G_2.
  ② (i) If every vertex in G_1 is adj. to every vertex in G_2: then since this is the minimum partition, G is complete.
  (ii) $\exists u\in G_1, v\in G_2$ s.t. u not adj. to v. So for each $\underset{\underset{G_1}{\uparrow}}{u_i}-\underset{\underset{G_2}{\uparrow}}{v_i}$ let $w_i:=u_i$ if $u_i\neq u$, otherwise $w_i=v_i$.
  Then $u,v\notin\{w_i\}$. So $\{w_i\}$ is a cut where $|\{w_i\}|=\lambda(G)$.

Chartrand & Harary Thm:

For any K, λ, δ with $0\leq K\leq\lambda\leq\delta$,

$\exists$ graph G s.t. $K(G)=K$
$\lambda(G)=\lambda$
$\delta(G)=\delta$



Pf: Let $H_1\cong K_{\delta+1}\cong H_2$.
$A:=\{u_1,\dots,u_K\}\subsetneqq V(H_1)$
$B:=\{v_1,\dots,v_\lambda\}\subsetneqq V(H_2)$
$F=\{u_1v_1,\dots,u_Kv_K,u_1v_{K+1},\dots,u_1v_\lambda\}$
Then $V(G)=V(H_1)\sqcup V(H_2)$
$E(G)=E(H_1)\sqcup E(H_2)\sqcup F$ satisfies the requirements.
(also, need to show that λ and K are indeed the minimum)
↖ show that if we use $\lambda'<\lambda$ or $K'<K$, then the graph must still be connected.

G is k-connected := $K(G)\geq k$

Separation: Given $u,v\in V(G)$, $S\subseteq V(G)\setminus\{u,v\}$,

Then S separates u & v := $\begin{cases} G-S \text{ disconnected} \\ u \,\&\, v \text{ are in different components of } G-S \end{cases}$

Cor: $\cdot$ $K(G)\leq |S|$

internally disjoint paths: paths that don't share any vertex except endpoints.

Menger's thm (max-flow min-cut): $\min\{|S| : S \text{ separates } u \,\&\, v\} = \max\{k : \exists k \text{ internally disjoint } u-v \text{ paths}\}$.

Pf: By induction on $v(G)$.

(so if we can find S where $|S|=k$ and k internally disjoint $u-v$ paths, then $K(G)=k$)

Whitney's thm: Given G with $n\geq 2$,

G is k-connected $\iff \forall u,v\in V(G)$ distinct, $\exists k$ internally disjoint $u-v$ paths.

Pf: "$\Leftarrow$": trivial.

"$\Rightarrow$": For non-adjacent u,v, use Menger. Otherwise remove uv edge and prove by contradiction.

Cor:
G is 2-connected
$\iff$ any $u,v\in V(G)$ or $e,f\in E(G)$ or $(v,e)\in V(G)\times E(G)$ lies on a common cycle.

G is **nonseparable** (or block) := G has no cut-vertex

$\iff G = K_2$ or G is 2-connected

H is a **block** of G := H is a maximal nonseparable subgraph of G. (i.e. H is a nonseparable subgraph of G that is not contained within any nonseparable subgraph of G)

$G_1 \cup G_2 := V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$, $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Block merging thm: If both G_1 and G_2 are k-connected and $|V(G_1) \cap V(G_2)| \geq k$ then $G_1 \cup G_2$ is k-connected

i.e. G_1 and G_2 share at least k common vertices

Cor: (1) Every block is an induced subgraph of G (i.e. B = subgraph of G induced by V(B)).

(2) Every two blocks have at most one vertex in common.

(3) If two blocks share a common vertex v, then v is a cut-vertex.

(4) The blocks of G partition E(G).

(5) Every cut-vertex belongs to at least two blocks.

Block-cut-vertex graph: Graph bc(G) where vertices are blocks or cut-vertices of original graph G,

and $\underset{\substack{\uparrow \\ \text{block} \\ \text{of } G}}{B}\underset{\substack{\uparrow \\ \text{cut-vertex} \\ \text{of } G}}{v} \in bc(G) \iff v \in V(B)$

Thm: G is connected $\iff$ bc(G) is a tree

Lemma: G is connected and has at least one cut-vertex $\Rightarrow$ G has at least two endblocks (i.e. blocks that are leaves in bc(G)).

Matching

Matching := set of edges that are pairwise non-adjacent

single vertex in matching := unmatched

saturated vertex in matching := matched

Maximal matching := a matching M not contained in any other matching.

$\iff$ any edge in $E(G) \setminus M$ is adjacent to some edge in M, and M is a matching

Maximum matching := matching with maximum size.

Perfect matching := matching where all vertices are separated $\Rightarrow |M| = \frac{n}{2}$ and n even.

M-alternating path := path that alternately uses edges in M and not in M.

M-augmenting path := M-alternating path with both ends being single vertices

Augmenting path theorem: M is a maximum matching $\iff$ there is no M-augmenting path (PF: by considering the set symmetric difference between M and any maximum matching)

$A \Delta B$:= set symmetric difference between A and B

Tutte thm: G has a perfect matching $\iff$ for any $S \subsetneq V(G)$, $\underset{\substack{\uparrow \\ \text{number of odd components}}}{o}(G-S) \leq |S|$

Bipartite matching

M is a **complete** matching from X to Y := every vertex in X is M-saturated, i.e. $|X| = |M|$

M is a **perfect** matching $\iff |X| = |Y| = |M|$

Hall's marriage thm := $\underset{\substack{\shortparallel \\ \langle X \sqcup Y, E(G) \rangle}}{G}$ has a complete matching $\iff \forall S \subseteq X$, $|S| \leq |\underset{\substack{\uparrow \\ \text{neighbours}}}{N}(S)|$

Given any nonempty sets $S_i \subseteq S$, $(1 \leq i \leq N)$, a system of distinct representatives exists

$\iff \exists$ complete matching $(S_1, \dots, S_N) \to S$

$\iff \forall \{S_{i_1}, \dots, S_{i_k}\}, |N(\{S_{i_1}, \dots, S_{i_k}\})| \geq k$

$A \subseteq V(G)$ is <u>independent</u> := any two vertices in A are non-adjacent

<u>max. independent set</u> := $\max\{|A| : A \text{ is independent}\} =: \alpha(G)$

$M \subseteq E(G)$ is a <u>matching</u> := any two edges in M are non-adjacent

<u>max. matching</u> := $\max\{|M| : M \text{ is a matching}\} =: \alpha'(G)$

$Q \subseteq V(G)$ is a <u>vertex cover</u> := every edge is incident to some vertex in Q

<u>min. vertex cover</u> := $\min\{|Q| : Q \text{ is a vertex cover}\} =: \beta(G)$

$W \subseteq E(G)$ is an <u>edge cover</u> := every vertex is adjacent to some edge in W.

<u>min. edge cover</u> := $\min\{|W| : W \text{ is an edge cover}\} =: \beta'(G)$

<u>Thms:</u>

- A is a maximum independent set $\Leftrightarrow V(G)\setminus A$ is a minimum vertex cover. $\therefore \alpha(G)+\beta(G) = v(G)$
- If Q is a vertex cover and M is a matching then $|M| \leq |Q|$
  - Cor: $M = Q \Rightarrow M$ is max matching & Q is min vertex cover
  - Cor: $\alpha'(G) \leq \beta(G)$
- If $G = C_{2k+1}$ then $\alpha'(G) < \beta(G)$
- G bipartite $\Rightarrow \alpha'(G) = \beta(G)$
- If G has no isolated vertices then $\alpha'(G) + \beta'(G) = v(G)$ (and hence $\alpha(G) \leq \beta'(G)$)

Graph Colouring

- <u>k-colouring</u> := Function $\theta : V(G) \to A$ where $|A| = k$ such that $uv \in E(G) \Rightarrow \theta(u) \neq \theta(v)$
- G is <u>k-colourable</u> := a k-colouring exists for G
- <u>Chromatic number</u> of G := $\chi(G) := \min\{k \in \mathbb{N} \mid G \text{ is k-colourable}\}$
- $\chi(G) = 1 \Leftrightarrow G$ is empty (i.e. no edges)
- G has order n and $\chi(G) = n \Leftrightarrow G = K_n$
- $\chi(G) = 2 \Leftrightarrow G$ is nonempty bipartite

- <u>Clique number</u> of G := $\omega(G)$:= order of largest complete subgraph in G
- $\omega(G) \leq \chi(G)$

- Given $G = C_n$:
  - $\chi(G) = 2 \Leftrightarrow n$ is even
  - $\chi(G) = 3 \Leftrightarrow n$ is odd
- H is a subgraph of $G \Rightarrow \chi(H) \leq \chi(G)$
- G has components $G_1, \dots, G_k \Rightarrow \chi(G) = \max\{\chi(G_1), \dots, \chi(G_k)\}$
- G has blocks $B_1, \dots, B_k \Rightarrow \chi(G) = \max\{\chi(B_1), \dots, \chi(B_k)\}$
- <u>Removing a vertex or edge</u>: $\forall v \in V(G): \chi(G) - 1 \leq \chi(G-v) \leq \chi(G)$. $\forall e \in E(G): \chi(G) - 1 \leq \chi(G-e) \leq \chi(G)$
- $\forall i \in A$, $\theta^{-1}(i)$ is an independent set
- k-colouring $\Leftrightarrow$ partitioning of vertices into k independent sets
- $\chi(G)\alpha(G) \geqslant n$

- <u>Greedy colouring algorithm</u>:
  - Order all vertices $v_1, v_2, \dots, v_n$
  - For each vertex v_i, assign it the least colour unused by its neighbours before it (i.e. $N(v_i) \cap \{1, \dots, i-1\}$)
- $\chi(G) \leq$ result of greedy colouring algorithm $\leq \Delta(G)+1$
- <u>Brooks</u>: G connected:
  $G \cong K_n$ or $G \cong$ odd cycle $\Leftrightarrow \chi(G) = \Delta(G) + 1$ (i.e. G not complete and not odd cycle $\Leftrightarrow \chi(G) \leq \Delta(G)$)
- <u>Thm</u>: $uv \notin E(G)$:
  $\{$k-colouring of $G\} = \{$k-colouring of $G+uv\} \sqcup \{$k-colouring of $G/uv\}$

  "contraction" of G on uv — unifies vertices u and v.
- $P_G(k)$:= number of k-colourings of G
- Given $uv \notin E(G)$: $P_G(k) = P_{G+uv}(k) + P_{G/uv}(k)$
- $k < \chi(G) \Leftrightarrow k < \chi(G+uv)$ and $k < \chi(G/uv)$ $\dashrightarrow$ $\chi(G) = \min\{\chi(G+uv), \chi(G/uv)\}$

G is critical $:=$ For any proper subgraph H of G, $\chi(H) < \chi(G)$

$\iff \forall e \in E(G)$, $\chi(G-e) = \chi(G) - 1$ (for G without isolated vertices only)

G is k-critical $:=$ G is critical and $\chi(G) = k$

Properties of critical graphs:
- no isolated vertices
- 2-connected

$G_1 + G_2 :=$ $V(G_1 + G_2) = V(G_1) \sqcup V(G_2)$

$E(G_1 + G_2) = E(G_1) \sqcup E(G_2) \sqcup \{uv : u \in G_1, v \in G_2\}$

Addition of critical graphs: G_1 is k_1-critical, G_2 is k_2-critical $\Rightarrow$ $G_1 + G_2$ is (k_1+k_2)-critical

– use the fact that any odd cycle is 3-critical to construct k-critical graphs of any order

Wedge: G_1 and G_2 nontrivial k-critical graphs $\Rightarrow$ $G_1 \wedge G_2$ is k-critical

$\nwarrow$ $G_1 \wedge G_2$ is any graph formed where G_1 & G_2 shares a single vertex v,
and $v_1 v \in G_1$ and $v_2 v \in G_2$.
and $G_1 \wedge G_2 = ((G_1 - v_1 v) \cup (G_2 - v_2 v)) + v_1 v_2$